Quantum Mechanics
ISI B.Math/M.Math
Midterm Exam : Feb 22,2025

Total Marks: 70
Time : 3 hours
Answer all questions

1.(**Marks : 5 + 5 + 4 = 14**)

(a) If two or more distinct(linearly independent) solutions to the time-independent Schrödinger equation have the same energy E, these states are said to be degenerate. Prove that in one dimension, there are no degenerate bound states. [Hint: Start by supposing that there are two solutions ψ_1 and ψ_2 with the same energy E. Try to show that $\psi_2 \frac{d\psi_1}{dx} - \psi_1 \frac{d\psi_2}{dx}$ is a constant, use the appropriate properties of bound state ψ at infinity to demonstrate that the solutions are not distinct and hence show a contradiction]

(b) Let $P_{ab}(t)$ be the probability of finding a particle of mass m in the range $(a < x < b)$ at time t. Show that

$$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t)$$

where $J(x,t) = \frac{i\hbar}{2m}\left(\Psi(x,t)\frac{\partial \Psi^*(x,t)}{\partial x} - \Psi^*(x,t)\frac{\partial \Psi(x,t)}{\partial x}\right)$ and $\Psi(x,t)$ is the wave function of the particle

(c) If $V(x)$ is an even function of x, then the solutions $\psi(x)$ of the one-dimensional time-independent Schrödinger equation can always be taken to be either even or odd.

2. (**Marks : 3 + 3 + 4 + 4 = 14**)

A one dimensional harmonic oscillator of mass m has potential energy $V(x) = \frac{1}{2}m\omega^2 x^2$. Consider the operators $a = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x + ip)$ and $a^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x - ip)$

(a) Express the Hamiltonian $\hat{H}$ in terms of a and $a^\dagger$

(b) Show that if ψ_n is a solution of the time independent Schrödinger equation with energy E_n, then $a\psi_n$ is a solution with energy $(E_n - \hbar\omega)$

(c) Given that $a^\dagger\psi_n = \sqrt{n+1}\psi_{n+1}$ and $a\psi_n = \sqrt{n}\psi_{n-1}$, find the expectation value of the potential energy $< V >$ in the state ψ_n.

(d) Show that the average kinetic energy $< T >$ is equal to the average potential energy $< V >$ for any energy eigenstate ψ_n of the harmonic oscillator. What is this value for the ground state of the harmonic oscillator ?

3. (**Marks : 6 + 4 + 4 = 14**)

A particle of mass m is moving in an infinite potential well of width a under the potential $V(x) = 0$ for $0 \leq x \leq a$ and $V(x) = \infty$ otherwise.

(a) Find the possible values of the energies E_n of the stationary states and the corresponding wave functions $\psi_n(x)$(solutions of the time independent Schrodinger equation)

(b) If the particle starts out in starts out in the left half of the well, and is at $(t = 0)$ equally likely to be found at any point in that region, what is the initial wave function $\Psi(x, 0)$? Is the particle in a state of definite energy ? Explain.(Assume it is real)

(c) With the particle in the initial state described in part (b) , what is the probability that a measurement of energy would yield the value $\frac{\pi^2\hbar^2}{2ma^2}$?

4. (**Marks : 4 + 5 + 5 = 14**)

A free particle has the initial wave function

$$\Psi(x, 0) = Ae^{-a|x|}$$

where A and a are real constants.

(a)Find A

(b) Find $\phi(k)$, the wave number k probability amplitude corresponding to the above initial state. If a measurement is made on this initial state at $t = 0$, what will be the most probable value of momentum found ?

(c) Construct $\Psi(x, t)$ in terms of an integral and discuss the limiting cases (a very large and very small) in terms of the spread in position and momentum. (No extra calculation is needed to discuss the limiting cases, only deduction of general behavior from the integral).

5. (**Marks : 2 + 6 + 6 = 14**)

(a) State the conditions on energy for finding bound state and scattering state solutions to the Schrodinger equation.

(b) A particle of mass m is moving under the potential $V(x) = -\alpha\delta(x)$ where α is a positive constant. Using appropriate boundary conditions show that it can have one and only one bound state and find its energy and corresponding wave function. Sketch the wave function.
(c) Now consider scattering states of the potential in (b). Show that the transmission coefficient is given by

$$T = \frac{1}{1+\beta^2}$$

where $\beta = \frac{m\alpha}{\hbar^2 k}$ and $k = \frac{\sqrt{2mE}}{\hbar}$. Discuss the difference with the classical picture.